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CONTROL OF THE RELATIVE MOTIONS OF A PENDULUM ON A ROTATING BASE†

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The problem of the control and optimization of the rotatory-oscillatory motions of a two-mass system, generalizing the model of a spherical pendulum, is investigated. The system contains a carrier body (a base) rotating about a vertical axis, with a plane pendulum attached to it. Particular is given attention to solving the problem of controlling the relative motions of the pendulum by regulating the velocity at which the base rotates. The relative stationary states of the pendulum are defined and studied. Near-time-optimal feedback controls are constructed in the neighbourhood of stable states. Near-optimal controlled motions are suggested and investigated in both oscillatory and rotatory modes. Also, the problem of a transfer from one of these modes to the other is considered. The qualitative properties of the controlled systems are established and analysed.

Numerous publications have been devoted to working out analytical and computational methods of solving classes of problems of the optimal and near-optimal control of motions in "pendulum-type" systems (see the monographs [1–3] and their bibliographies). Rotatory-oscillatory motions may form a basis of some process and/or constitute perturbations in the control of the motions of complex dynamical systems. Among these are manipulation robots, including those of the recuperation type, aircraft with elastic components of substantial length, crane systems and helicopters transporting swinging loads on flexible suspensions of various kinds, etc. Among the more frequently employed controls are inertial exitations (forces and moments of reaction forces) caused by controlled displacements of a relative equilibrium position of the rotatory-oscillatory system. Below we will investigate problems of the dynamics and control of the motions of a plane pendulum attached to a base rotating about a vertical axis.

1. FORMULATION OF THE PROBLEM

Consider the motions of a two-mass system consisting of a rigid body (base) with an attached pendulum (see Fig. 1). The body can rotate freely about the vertical Z axis of an inertial system of coordinates XYZ. The pendulum (or rotator) performs relative plane oscillations or rotations about an axis passing through the vertical Z axis and orthogonal to it.

Expressions for the kinetic energy K and potential energy Π of the system can be calculated by standard methods, and may then be used to write down Lagrange's equations of motion

$$\begin{gathered}
\mathrm{K} = \tfrac{1}{2} I(\theta)\dot\varphi^2 + \tfrac{1}{2} ml^2\dot\theta^2 + mld\dot\varphi\dot\theta\cos\theta, \quad \Pi = mgl(1-\cos\theta) \\
(d/dt)[I(\theta)\dot\varphi + mld\dot\theta\cos\theta] = M_\varphi, \quad I(\theta) = I_0 + m(d^2 + l^2\sin^2\theta) \\
ml^2\ddot\theta + mld\ddot\varphi\cos\theta - ml^2\dot\varphi^2\sin\theta\cos\theta + mgl\sin\theta = M_\theta
\end{gathered} \tag{1.1}$$

where φ is the angle of rotation of the rigid body and I_0 is its moment of inertia, θ is the angular variable of the pendulum, of mass m and length l, and d is the distance from the axis of rotation of the base to the suspension point of the pendulum (the "arm"). The function $I(\theta)$ has the meaning of the reduced instantaneous moment of inertia of the system about the Z axis, it satisfies the conditions $I_0 + md^2 \leqslant I(\theta) \leqslant \leqslant I_0 + m(d^2 + l^2)$. In the special case where $I_0 = 0$ and $d = 0$ we obtain the well-known classical model of a spherical pendulum. By (1.1), the system is subject to torques M_φ and M_θ belonging to some class of functions.

If $M_\varphi \equiv 0$, the angular momentum about the vertical Z axis is conserved, that is, system (1.1) has an area integral

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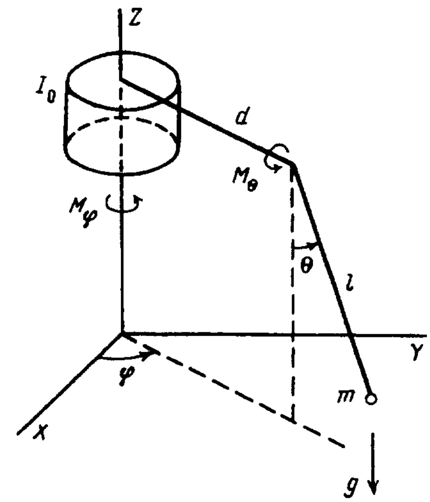

Fig. 1.

$$I(\theta)\dot\varphi + mld\dot\theta\cos\theta = c = \text{const} \tag{1.2}$$

for any M_θ, as follows from the first equation. Equation (1.2) can be used to investigate the second equation. In particular, if $M_\theta = A(\theta) + B(\theta)\,\dot\theta^2$, it can be completely integrated analytically, since it can be reduced to a Bernoulli-type equation

$$\frac{d}{dt}(J(\theta)\dot\theta) + D(\theta)\dot\theta^2 + F(\theta) = 0 \tag{1.3}$$

If the torque M_θ is such that $D(\theta) = -1/2J'(\theta), F(\theta) = W'(\theta)$, where J and W are 2π-periodic functions of θ, then (1.3) is the equation of a generalized pendulum [2, 3]. It admits of an energy integral $E = 1/2J\dot\theta^2 + W$, where J and W (the generalized moment of inertia and potential energy, respectively) are functions of θ and c which are known from (1.1) and (1.2).

The torques M_φ and M_θ may be regarded as controls to be chosen so as to steer system (1.1) to some fixed phase state, and minimize some performance index taking into account constraints of various types. With this general formulation of the optimal control problem one cannot obtain any really meaningful results. We will therefore restrict the formulation by assuming that one of the generalized coordinates, φ or θ, varies in a prescribed way, considering their derivatives as control functions and assuming that there are no external torques M_θ or M_φ. This yields restricted formulations of control problems for the variables θ or φ, where the control is implemented by kinematic (inertial) means. Thus, if the variable $\dot\theta$ is taken as the control υ, then, $M_\varphi \equiv 0$, we have a family of controlled systems of the following form (c is the parameter of the family)

$$\dot\varphi = I^{-1}(\theta)(c - mldv\cos\theta), \quad \dot\theta = v, \quad v \in V \tag{1.4}$$

The case $c = 0$ is of special interest. For system (1.4), one can formulate problems of varying the variables φ and θ within certain limits. The control υ can be implemented by an electric motor with a reduction gear of high gear ratio, located at the pendulum joint [2, 3].

We will consider the basic and much less trivial formulation of the problem in which the variable φ is controlled kinematically and θ varies as in (1.1) with $M_\theta \equiv 0$. We then have a control system of the form

$$\begin{gathered}\ddot\theta - \omega^2\sin\theta\cos\theta + \nu^2\sin\theta = -\varkappa u\cos\theta \\ \dot\omega = u, \quad \dot\varphi = \omega \quad (\nu^2 = g/l, \ \varkappa = d/l)\end{gathered} \tag{1.5}$$

According to (1.5), the angular acceleration $\ddot\varphi = u$ of rotation of the base is taken to be the control variable. The efficiency of this control for θ strongly depends on $\varkappa$—the normalized length of the arm d. It should be noted that the variable φ does not occur in Eqs (1.5). This enables us to pose a number of control problems for the relative state of the pendulum, ignoring the angular orientation of the body. For applications it is very important to investigate the relative equilibrium positions θ_e of the pendulum at $u \equiv 0$ (ω = const), and to formulate and solve the optimal control problem for motions of the pendulum in the neighbourhood of the above-mentioned stationary points of the system $\theta = \theta_c(\omega^2)$, $\dot{\theta} = 0$, $\dot{\varphi} = \omega$, which correspond to stable states; see below.

2. DETERMINATION AND INVESTIGATION OF THE RELATIVE EQUILIBRIUM POSITIONS OF THE PENDULUM

In order to reduce the number of parameters in system (1.5), we will introduce a non-dimensional time variable $t' = \upsilon t$ and the change variables: $\omega' = \omega/\upsilon$, $u' = u/\upsilon^2$. Omitting primes for brevity, we obtain equations with $\upsilon = 1$. Let us consider the case in which the angular velocity is kept constant, $\dot{\varphi} = \omega$ ($u \equiv 0$). Then the first equation for θ in (1.5) admits of the energy integral E:

$$E = \tfrac{1}{2}\dot{\theta}^2 + U(\theta, \omega^2), \quad -\pi < \theta \leqslant \pi \pmod{2\pi}, \quad 0 \leqslant \dot{\theta}^2, \quad \omega^2 < \infty$$
$$U(\theta, \omega^2) = 1 - \cos\theta - \tfrac{1}{2}\omega^2 \sin^2\theta, \quad -\infty < U \leqslant 2, \quad \min_\theta U \leqslant E < \infty \tag{2.1}$$

where U is the potential energy of oscillations or rotations of the pendulum about the regularly rotating base. Since the function U is 2π-periodic with respect to θ and symmetric about $\theta = 0$, it will suffice to represent it in the interval $0 \leqslant \theta \leqslant \pi$. Figure 2 illustrates the function $U(\theta)$ for different ω values. It follows from the expression for U in (2.1) that for $0 \leqslant \omega^2 \leqslant 1$ the function has a unique minimum and maximum:

$$\min_\theta U(\theta, \omega^2) = U(0, \omega^2) \equiv 0, \quad \max_\theta U(\theta, \omega^2) = U(\pi, \omega^2) \equiv 2$$
$$U'(0, \omega^2) = 0, \quad U''(0, \omega^2) = 1 - \omega^2, \quad U'''(0, 1) = 0 \tag{2.2}$$
$$U^{IV}(0, 1) = 3; \quad U'(\pi, \omega^2) = 0, \quad U''(\pi, \omega^2) = -1 - \omega^2, \quad 0 \leqslant \omega^2 \leqslant 1$$

It can be verified by differentiation that $U > 0$ in a small neighbourhood of the singular point $\theta_e = 0$ ($\theta \neq 0$), which is a centre, if $\omega^2 < 1$; $U < 2$ if $\theta \neq \pi$ in a small neighbourhood of the singular point $\theta_e = \pi$, which is a saddle point, for all $0 \leqslant \omega^2 \leqslant 1$, that is, the minimum and maximum (2.2) are

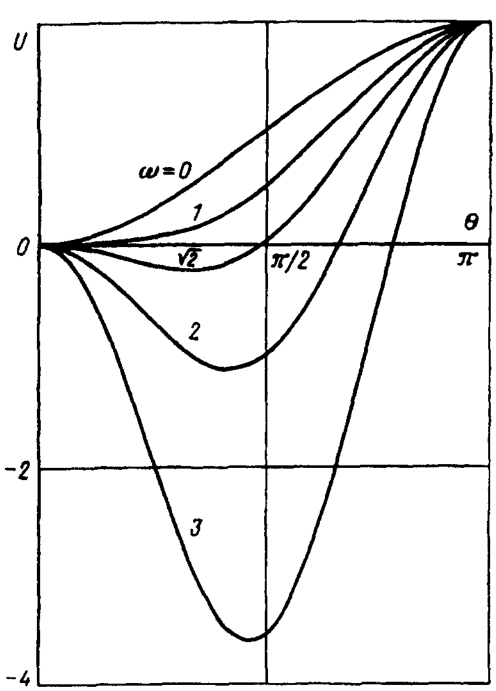

Fig. 2.

strict. Thus, if the body is set in motion at a velocity $\dot{\varphi} = \omega$, $\omega^2 \leq 1$ (i.e. $\omega^2 \leq \upsilon^2$), provided that $\theta = \dot{\theta} = 0$, the system will perform stable rotation as a whole relative to the variables $\dot{\omega}$, θ, $\dot{\theta}$; the relative equilibrium position $\theta_e = \pi$, $\theta = 0$ will be exponentially unstable: the variation will be

$$\delta\theta \sim \delta\theta^0 \exp(\sqrt{1+\omega^2}\, t)$$

Let us consider the relative equilibrium positions when $\omega^2 > 1$, that is, $\omega^2 > \upsilon^2$ in dimensional variables. It than follows from an analysis of the function $U(\theta, \omega^2)$ of (2.1) that the stable (lower) equilibrium position $\theta_e = 0$ splits into two equilibrium: $\theta_e = 0$ and $\theta_e = \theta_* = \arccos\omega^{-2}$; the equilibrium $\theta_e = \pi$ is preserved (see Fig. 2). Simple analysis indicates that the quantities $\theta_e = 0, \pi$ (saddle points) correspond to strict maxima, while $\theta_e = \theta_*$ (a centre point) corresponds to a strict minimum, and moreover

$$U_*(\omega^2) = U(\theta_*, \omega^2) = -\tfrac{1}{2}(\omega^2 + \omega^{-2}) + 1 < 0, \quad U''(\theta_*, \omega^2) = \omega^2 - \omega^{-2} > 0 \tag{2.3}$$

Analogous statements hold for uniform rotation of the system as a whole in the state $\dot{\varphi} = \omega$, $\dot{\theta} = 0$, $\theta_e = \theta_*(\omega^2)$, 0, π. Note that the minimum function satisfies the relation $U_*(\omega^2) \approx -1/2\omega^2$ if $\omega^2 \gg 1$, that is, it decreases fairly rapidly as $|\omega|$ increases where $\theta_*(\omega^2) < \pi/2$, $\theta_* \to \pi/2$, as $\omega^2 \to \infty$, but $U'(\pi/2, \omega^2) = 1$ irrespective of ω.

We will now investigate the orbits in the phase plane $(\theta, \dot{\theta})$ for different values of ω^2 and E in accordance with (2.1). Figure 3 illustrates fragments of phase orbits for $\omega = 3$ and different E values. Owing to symmetry, we can confine our attention to the first quadrant. We will first consider the simple situation $E > 2$, corresponding to the mode of relative rotations; we have

$$\begin{aligned} &\dot{\theta} = \pm\eta(\theta, \omega^2, E), \quad \eta(\theta, \omega^2, E) = \sqrt{2}[E - U(\theta, \omega^2)]^{1/2}, \quad E > 2 \\ &\min_\theta \eta = \sqrt{2}(E-2)^{1/2}, \quad \max_\theta \eta = \sqrt{2E}, \quad 0 \leq \omega^2 \leq 1 \\ &\min_\theta \eta = \sqrt{2}(E-2)^{1/2}, \quad \mathrm{extr}_\theta \eta = \sqrt{2E}, \quad \max_\theta \eta = \sqrt{2}[E - 1 + \tfrac{1}{2}(\omega^2 + \omega^{-2})]^{1/2}, \omega^2 > 1 \end{aligned} \tag{2.4}$$

When $E > 2$ the pendulum performs monotonic relative rotations at a positive velocity $\theta = \eta$ or

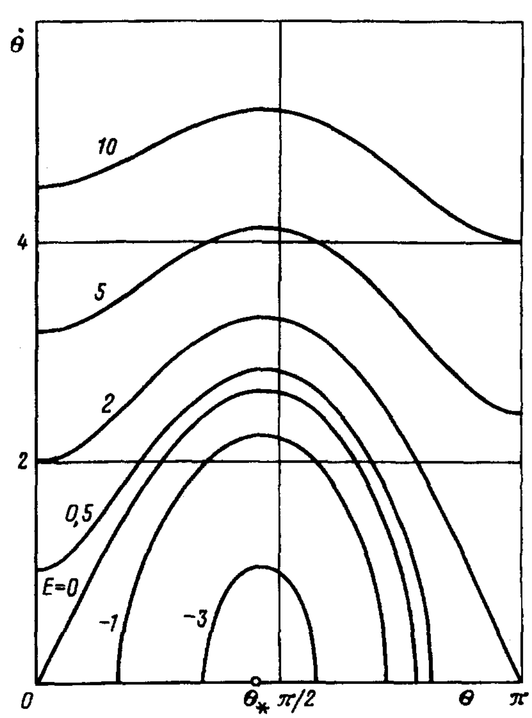

Fig. 3.

negative velocity $\dot{\theta} = -\eta$ whose magnitude oscillates between certain limits defined by (2.4) (see Fig. 3). In the limit as $E \downarrow 2$ we obtain a separatrix; for values of $0 \leqslant \omega^2 \leqslant 1$ it corresponds qualitatively to the classical case $\omega^2 = 0$ – *a* pendulum with fixed axis [1–3]. Significant differences are observed when $\omega^2 > 1$: a local minimum appears at $\theta = 0$ and a maximum at $\theta = \theta_*(\omega^2)$ ($\theta_* \approx \pi/2$, $\omega^2 \gg 1$). Their values can be computed by (2.4); when $E = 2$ and $\omega^2 \gg 1$ we have an asymptotic form $\max_\theta \eta \sim |\omega|$ (see Fig. 3).

We will now investigate the phase orbits in oscillatory motion, using formulae (2.4) for $\dot{\theta}$, η with $E < 2$. The nature of the oscillations depends essentially on the values of the parameters ω^2 and E. We will first consider the case of a system qualitatively similar to the classical pendulum (see above). The phase orbits and turning points are defined analytically in the form

$$\begin{aligned}
&\dot{\theta} = \pm\eta(\theta, \omega^2, E), \quad \theta^- \leqslant \theta \leqslant \theta^+, \quad \theta^- = -\theta^+, \quad 0 \leqslant E < 2, \quad 0 \leqslant \omega^2 \leqslant 1 \\
&\theta^+ = \theta^+(\omega^2, E) = \arccos z(\omega^2, E), \quad 0 \leqslant \theta^+ > \pi, \quad -\pi < \theta^- \leqslant 0 \\
&z = \omega^{-2}(1 - [(1-\omega^2)^2 + 2\omega^2 E]^{1/2}), \quad -1 > z \leqslant 1
\end{aligned} \tag{2.5}$$

By (2.5), the relative oscillations of the pendulum occur within symmetrical limits $\theta^{\pm}$. The unique stationary point, a centre point, $\theta_e = 0$, corresponds to $E = 0$. The orbits are convex either upward ($\dot{\theta} = \eta$) or downward ($\dot{\theta} = -\eta$) (see Fig. 3).

A somewhat more complicated phase portrait is observed when $\omega^2 > 1$. If $0 \leqslant E < 2$, formulae (2.5) remain true, but a qualitative change occurs in the behaviour of the orbits: η is convex upward (concave) at the ends of the interval (θ^-, θ^+) and convex downward in the middle part of the interval. In the limit as $E \downarrow 0$ the system has a saddle point and the phase orbit has a shape resembling the symbol ∞. Naturally, the oscillations of the pendulum for $0 < E < 2$ occur within the symmetrical limits $\theta^{\pm}$ according to formulae (2.5) (see Fig. 3).

If $E < 0$, the domain of oscillations splits symmetrically into two (see Figs 2 and 3). The oscillations of the pendulum may occur within the limits $\theta \in [\theta^-, \theta^+] \subset (0, \pi)$ or $\theta \in [\theta^-, \theta^+] \subset (-\pi, 0)$. To fix our ideas, let us consider the case of positive limits. The formulae of type (2.5) for the phase orbit and stationary points take the form

$$\begin{aligned}
&\dot{\theta} = \pm\eta(\theta, \omega^2, E), \quad \theta^- \leqslant \theta \leqslant \theta^+, \quad 0 > E \geqslant U_*(\omega^2) = 1 - \tfrac{1}{2}(\omega^2 + \omega^{-2}) \leqslant 0 \\
&\theta^{\pm}(\omega^2, E) = \arccos z^{\mp}(\omega^2, E), \quad 0 < \theta^- < \pi/2, \quad \theta^- < \theta^+ < \pi \\
&z^{\pm} = \omega^{-2}(1 \pm [(1-\omega^2)^2 + 2\omega^2 E]^{1/2}), \quad -1 < z < 1
\end{aligned} \tag{2.6}$$

The stationary point θ_* for a fixed value of ω, $|\omega| > 1$, is defined by (2.3) and (2.6): $\theta_*(\omega^2) = \arccos\omega^{-2}$, and $\theta_* \uparrow \pi/2$ as $\omega^2 \to \infty$, corresponding to mechanical considerations. The oscillations occur within limits $\theta^- \leqslant \theta \leqslant \theta^+$ which are asymmetrical about $\theta_e = \theta_*$; this follows from formulae (2.6) for θ_*, $\theta^{\pm}$. It is interesting to observe that $0 < \theta^- < \pi/2$, $0 < \theta^+ < \pi$, but $\theta^+ \gtrless \pi/2$. We have thus completed our investigation of the phase portrait of relative oscillations of the pendulum when the base is rotating uniformly, which will be needed in what follows.

The previous analysis implies a natural and non-trivial formulation of the optimal control problem for system (1.5): to bring the pendulum to a state of relative stable equilibrium θ_e corresponding to a fixed velocity of rotation ω of the base. The initial state θ^0, $\dot{\theta}^0$, ω^0 of the system may be arbitrary; the control—the angular acceleration of the base—is taken from a given class of functions $\dot{\omega} = u \in \{u\}$, such as piecewise-continuous bounded functions $|u| \leqslant u_0$, and the performance index is the time of the process [1–4]. It is of some interest to bring the pendulum to a state of rotation $E > 2$ or to a state of oscillation with fixed total energy E: $U_*(\omega^2) < E < 2$. The construction of exact solutions of such optimal control problems for non-linear oscillatory systems of type (1.4), (1.5) meets considerable difficulties. We therefore propose the use of approximate analytical methods of perturbation theory, which involve introducing a small parameter [1–3].

3. APPROXIMATE INVESTIGATION OF "INERTIAL CONTROL" PROBLEMS

Let us consider the restricted formulations of problems of control by kinematic adjustment of one of the angular variables θ or φ, as formulated in Sections 1 and 2.

Rotation of the base. Let us investigate the possibility of controlling the rotations of the rigid body (the base), as described by Eqs. (1.4), by varying the angular velocity of the pendulum. The time-optimal problem for system (1.4) is

$$\varphi(0)=\varphi^0,\quad \theta(0)=\theta^0,\quad \varphi(t_f)=\varphi^f (\operatorname{mod} 2\pi),\quad t_f \to \min_\upsilon,\quad |\upsilon| \leqslant \upsilon_0 \tag{3.1}$$

The system is controllable in the sense of (3.1) if $c \neq 0$; in the limiting case of $c = 0$ it is not always controllable, since by (1.4) the variables φ and θ satisfy the following relation for any $\upsilon \not\equiv 0$:

$$\varphi-\varphi^0=-\varkappa a^{-1}(\operatorname{arctg} s-\operatorname{arctg} s^0),\quad a=(I_0/(ml^2)+\varkappa^2)^{1/2},\quad s=a^{-1}\sin\theta \tag{3.2}$$

where we can put $\varphi^0 = 0$ without loss of generality. Since the orientation φ^f is defined by (3.1) to within 2π, it follows from (3.2) that the condition for controllability is

$$|\varphi^f| \leqslant 2\varkappa a^{-1}\operatorname{arctg} a^{-1} \quad (|\varphi^f|<\pi) \tag{3.3}$$

This expression has an obvious interpretation in terms of mechanics. We note only that the right-hand side of (3.3) is bounded by π for all values of I_0, ml^2, md^2, since $\varkappa a^{-1} \leqslant 1$ and $a^{-1} < \infty$; the limiting value is reached if I_0, $d \to 0$, that is, we have a spherical pendulum. In addition, when the required orientation φ^f is reached the body will remain in that angular position if we take $\upsilon \equiv 0$ for $t > t_f$. It is also interesting to observe that condition (3.3) does not depend on υ_0. A time-optimal control υ^*, in the open-loop and feedback forms, can be constructed by carrying out the following elementary operations

$$\begin{gathered}\int_0^{t_f} \upsilon^*(t)dt=\pm\upsilon_0 t_f^{\pm}=\arcsin a\frac{s^0-\operatorname{tg}\psi^f}{1+s^0\operatorname{tg}\psi^f}-\theta^0\\ 0<t_f^*=\min(t_f^+,t_f^-),\quad s^0=a^{-1}\sin\theta^0,\quad \psi^f=\varphi^f a/\varkappa\end{gathered} \tag{3.4}$$

If the controllability condition (3.3) is satisfied, problem (3.4) has a solution.

If $c \neq 0$, system (1.4), as already noted, is controllable in the sense of (3.1). If $\varepsilon = |c|(mld\upsilon_0)^{-1} \ll 1$, the solution of the problem in the general case (when condition (3.3) holds with a safety margin) is close to that described above. It can be constructed approximately by methods of perturbation theory [1–3] with a small parameter ε.

If the control applied to the body is small, that is, $\varepsilon^{-1} = mld\upsilon_0|c|^{-1} \ll 1$, but $\mu = \upsilon_0(I_0 + md^2)|c|^{-1} \sim 1$, then the approximate methods of optimal control developed for regularly perturbed systems [1–3] are also applicable. In the first approximation, the control υ^* must lead to a decrease in $I(\theta)$, that is, to the minimization of $\sin^2\theta$; this gives the result $\upsilon^* \approx -\upsilon_0\sin 2\theta$.

But if the control υ is also small: $\mu \sim \varepsilon \ll 1$, then it is natural to apply locally optimal control laws [1], such as

$$\upsilon^*=-\upsilon_0\operatorname{sign}[(\varphi^f-\varphi)\cos\theta+k^2\theta]\quad(\varphi^f-\varphi,\theta;\operatorname{mod}2\pi) \tag{3.5}$$

If $k^2 = 0$, expression (3.5) for υ^* corresponds to a locally optimal mode in which the functional $J_\varphi = (\varphi^f - \varphi(t_f))^2$ is minimized, while if $k^2 > 0$ it corresponds to the minimization of $J_\varphi + k^2\theta^2(t_f)$. The parameter k^2 in (3.5) is a weighting factor, selected by numerical experimentation.

The case when $\varepsilon \sim 1$ but $\mu \gg 1$ corresponds to an asymptotically long time interval $t_f \sim \varepsilon^{-1}\upsilon_0^{-1}$. In the slow time τ one has virtually discontinuous (Bang-bang) variation of the angular variable θ, since $d\theta/d\tau = \upsilon|c|^{-1}(I_0 + md^2) \gg 1$. Approximate control is reduced to bringing the pendulum to the angular position $\theta = 0, \pi$ in a relatively short time $\Delta\tau \leqslant (\pi/2)|c|[\upsilon_0(I_0 + md^2)]^{-1} \ll 1$; one then puts $\upsilon \equiv 0$. The body plus pendulum rotates at maximum velocity $\dot\varphi \approx c(I_0 + md^2)^{-1}$, but until the required angular velocity φ^f (mod 2π) is reached the rotation velocity may be reduced by the quantity $mld\upsilon_0$ or made equal to zero if $|c| \leqslant mld\upsilon_0$.

The complete investigation of the general case, when $|c|(I_0\upsilon_0)^{-1} \sim mldI_0^{-1} \sim 1$, meets difficulties. Acceptable results in that case can be obtained by using a "penalty" method analogous to (3.5) or by numerical methods.

Control of the oscillations of the pendulum. On the basis of the analysis in Section 2 of the phase portrait of the oscillations of the pendulum ($E < 2$), we will investigate a number of control problems for motions in the neighbourhood of stable relative equilibrium positions. By (2.5), if $0 \leqslant \omega^2 \leqslant 1$, then $\theta = \dot\theta = 0$ is a stable position. For effective investigation of the control process on the basis of the non-dimensional equations (1.5), we introduce a small parameter ε characterizing the neighbourhood of the stationary point under consideration and the smallness of the control, so that it becomes possible to take into account the non-linear terms of the equation in the following way [2, 3]

$$\begin{gathered}\theta=\sqrt{\varepsilon}\alpha,\ \alpha \sim 1,\ \omega=\omega^*+\sqrt{\varepsilon}\gamma,\ \gamma\sim 1,\ \dot\omega=\varepsilon^{3/2}u,\ 0\leqslant\omega^*<1\\ \ddot\alpha+\Omega^2\alpha=-\varepsilon f\alpha^3-\varepsilon\varkappa u+O(\varepsilon^2),\ f\equiv \tfrac{1}{6}(-1+4\omega^{*2}+8\sqrt{\varepsilon}\omega^*\gamma)\\ \dot\gamma=\varepsilon u,\ u_1\leqslant u\leqslant u_2,\ u_{1,2}\sim 1,\ \Omega^2=1-\omega^{*2}-2\sqrt{\varepsilon}\omega^*\gamma-\varepsilon\gamma^2>0\end{gathered} \tag{3.6}$$

Furthermore, we assume that $\varkappa, \Omega^2 \sim 1$. We will formulate the problem of the time-optimal variation of the amplitude of small relative oscillations of the pendulum and the angular velocity of rotation of the base. To reduce Eqs (3.6) to the standard form of a controlled system with rotating phase, we make the change of variables $(\alpha, \dot\alpha) \to (a, \psi)$ [1–3] and obtain

$$\begin{gathered}\alpha=a\sin\psi,\ \dot\alpha=a\Omega\cos\psi,\ \gamma\equiv\gamma\ (a,\Omega,\gamma\sim 1)\\ \dot a=-(\varepsilon/\Omega)(\varkappa u+f(\gamma,\omega^*,\varepsilon)a^3\sin^3\psi)\cos\psi-a(\dot\Omega/\Omega)\cos^2\psi\\ \dot\gamma=\varepsilon u,\ \dot\psi=\Omega+O(\varepsilon),\ \dot\Omega=O(\varepsilon^{3/2})\end{gathered} \tag{3.7}$$

The performance index of the control, characterizing the optimal response time, as well as the initial and final conditions, are taken as follows:

$$t_f\to\min_u,\ u_1\leqslant u\leqslant u_2,\ a(0)=a^\circ,\ \gamma(0)=\gamma^\circ,\ a(t_f)=a^f,\ \gamma(t_f)=\gamma^f \tag{3.8}$$

In particular, if $a^f = 0$, the pendulum is brought to a state of relative equilibrium θ^f corresponding to the base rotating at velocity $\omega^f = \omega^* + \sqrt{\varepsilon}\gamma^f$; if $\gamma^f = 0$, the final velocity is $\omega^f = \omega$. We can assume, without loss of generality, that $a^f = \gamma^f = 0$. An approximate solution of problem (3.7), (3.8) is constructed using the maximum principle [4] and the method of averaging [5, 6]. Using asymptotic methods of optimal control [1–3] we obtain the following expressions, with relative error $O(\sqrt{\varepsilon})$ in terms of the functional and the orbit (ignoring $O(\varepsilon^{3/2})$:

$$\begin{gathered}u^*=\frac{u_1+u_2}{2}+\frac{u_2-u_1}{2}\begin{cases}\operatorname{sign}[\Delta a(k-\cos\psi)], & \Delta a\neq 0,\ |k|<1\\ \operatorname{sign}\Delta\gamma, & \Delta a=0,\ |k|\geqslant 1\end{cases}\\ k=\arg_k\left[\frac{\arcsin k-\sigma}{(1-k^2)^{1/2}}-\lambda\right],\ \lambda=\frac{\varkappa\Delta\gamma}{\Omega_0\Delta a},\ \sigma=\frac{\pi}{2}\frac{u_1+u_2}{u_2-u_1},\ \Omega_0=(1-\omega^{*2})^{1/2}\\ t_f=\frac{\pi}{\varepsilon}\frac{\varkappa^{-1}\Omega_0|\Delta a|+\Delta\gamma}{(\pi/2)(u_1+u_2)+(u_2-u_1)((1-k^2)^{1/2}-\arcsin k)},\\ a(t)=a^\circ+\Delta a\frac{t}{t_f},\ \gamma(t)=\gamma^\circ+\Delta\gamma\frac{t}{t_f}\\ \lambda\gtreqless\arg_\lambda(\sigma+\arcsin\lambda^{-1}+(\lambda^2-1)^{1/2}),\ \sigma\lesseqgtr\mp\pi/2,\ k(-\lambda,-\sigma)=-k(\lambda,\sigma)\end{gathered} \tag{3.9}$$

To construct a time-optimal control in the open-loop or feedback form, according to (3.9), one has to solve a transcendental equation in $k = k(\lambda, \sigma)$ using numerical or approximate analytical methods. A graphical solution is shown in Fig. 4. Note that the parameter λ is determined by the initial (current) values of the phase variables a and γ. The equation is solvable—in fact, uniquely—for all λ if $|\sigma| < \pi/2$, that is, $u_1 < 0$, $u_2 > 0$, a condition which is usually satisfied in real control systems; usually, $\sigma = 0(u_1 = -u_2)$. If $|\sigma| < \pi/2$, that is, $u_{1,2} > 0$, $u_{1,2} < 0$, the solution (if it exists) is not unique and one has to choose one of the roots $k_{1,2}$ that minimize t_f (see (3.9) and the curve $\sigma = -2$ in Fig. 4).

Thus, by algorithm (3.9), a weak control εu will steer system (3.6) from an arbitrary point a°, γ° into the $\sqrt{\varepsilon}$-neighbourhood of the point a^f, γ^f in an asymptotically long optimal time $t_f \sim \varepsilon^{-1}$. The phase

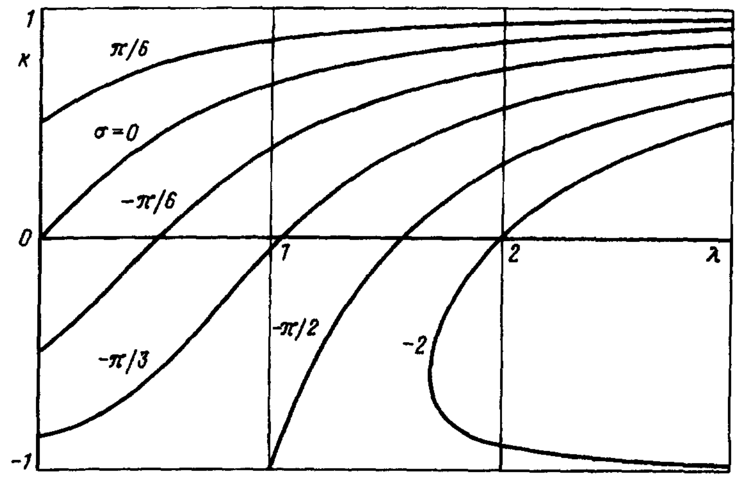

Fig. 4.

point of the original system (1.5), under the control $\dot\omega = \varepsilon^{3/2}u$, will be transferred from the $\sqrt{\varepsilon}$-neighbourhood of the required value

$$\theta^f = \theta_e + \sqrt{\varepsilon}\alpha^f,\quad \dot\theta^f = \sqrt{\varepsilon}\dot\alpha^f,\quad \omega^f = \omega^* + \sqrt{\varepsilon}\gamma^f$$

to an ε-neighbourhood. A solution analogous to (3.9) is obtained in the formulation of the control problem with the following scales introduced

$$\theta = \varepsilon\alpha,\quad \omega = \omega^* + \varepsilon\gamma,\quad \dot\omega = \varepsilon$$

This problem is equivalent to assuming (3.6) and representing $\dot\omega = \varepsilon u$ with the transformation $\sqrt{\varepsilon} \to \varepsilon$; however, with the problem normalized in this way, non-linear terms $O(\alpha^3)$ are not taken into consideration [2, 3].

We will now consider an optimal control problem for $\omega^2 > 1$, subject to assumptions similar to (3.6) but with $\dot\omega = \varepsilon^2 u$, $u \sim 1$. For convenience, we transform $\sqrt{\varepsilon} \to \varepsilon$ (see above). This yields a quasi-linear controlled system of type (3.7)

$$\begin{gathered}\ddot\alpha + \Omega^2\alpha = \varepsilon f(\alpha,\gamma) - \varepsilon\varkappa u/\omega^{*2},\quad \dot\gamma = \varepsilon u,\quad \Omega^2 = \omega^{*2} - 1/\omega^{*2}\\ f = (\tfrac{1}{2}\sin\theta_* - \omega^{*2}\sin 2\theta_*)\alpha^2 + 2\omega^*\gamma\cos 2\theta_*,\quad \cos\theta_* = 1/\omega^{*2}\end{gathered} \tag{3.10}$$

The approximate solution of the time-optimal problem (3.8) for system (3.10), with relative error $O(\varepsilon)$, is given by formulae (3.9), in which we substitute

$$\Omega_0 \to \Omega = (\omega^{*2} - 1/\omega^{*2})^{1/2},\quad \varkappa \to \varkappa/\omega^{*2}$$

(see the formulae for λ and t_f). Note that the occurrence of non-linear terms εf in (3.6), (3.7) and (3.10) does not affect the solution in the first approximation, since they disappear on averaging.

Control of the oscillations of the pendulum for rapid rotations of the base. Let us investigate an analogous problem, in the case where the rigid body is rotating at an asymptotically large angular velocity: $\omega^2 \gg 1$, that is, $\omega^2 \gg \nu^2 = g/l$ in dimensional variables. As established in Section 2, the stable stationary points are $\theta_* \approx \pi/2$, since $\theta_* = 1/\omega^{*2}$ by (2.3). We will investigate the oscillations of the pendulum in a small neighbourhood of the equilibrium position $\theta_e = \theta_*(\omega^{*2})$; to within an error $O(1/\omega^{*4})$, we have

$$\begin{gathered}\theta = \theta_* + \alpha/\omega^{*2},\quad \omega = \xi\omega^*,\quad \xi \sim 1,\quad \omega^{*2} \gg 1,\quad \varkappa \sim 1\\ \ddot\alpha + \xi^2\alpha = -(\varkappa/\omega^{*2})(1-\alpha)u,\quad \dot\xi = u/\omega^{*2},\quad u \sim 1\end{gathered} \tag{3.11}$$

(the dots denote differentiation with respect to the fast phase τ of rotations of the base: $\tau = \omega^* t$). For a weakly controllable ($\varepsilon = 1/\omega^{*2} \ll 1$) non-linear system, we can use asymptotic methods [1–3] to construct an approximate solution, after reducing the system to standard form with rotating phase, as in the case of (3.7). The strong non-linearity of system (3.11) is due to the essential dependence of the oscillation frequency $\Omega = \xi$ on the controlled slow variable ξ [2, 3]. Introducing the small parameter $\varepsilon = 1/\omega^{*2}$ for convenience and changing from variables $(\alpha, \alpha^{\bullet})$ to (a, ψ), we obtain relations for the controlled system in standard form and corresponding initial and final conditions:

$$\begin{aligned}
&\alpha = a\sin\psi,\ \alpha^{\bullet} = a\xi\cos\psi,\ \psi^{\bullet} = \xi + O(\varepsilon)\\
&a^{\bullet} = -\varepsilon\xi^{-1}h(a,\psi)u,\ h \equiv \varkappa(1 - a\sin\psi)\cos\psi + a\cos^2\psi,\ a(0) = a^0 > 0 \qquad (3.12)\\
&\xi^{\bullet} = \varepsilon u,\ \xi(0) = \xi^0 = \omega^0/\omega^*,\ u = \tfrac{1}{2}u^+ + \tfrac{1}{2}u^- w,\ u_1 \leqslant u \leqslant u_2\\
&a(\tau_f) = a^f \geqslant 0,\ \xi(\tau_f) = \xi^f;\ \tau_f = \min_w,\ |w| \leqslant 1
\end{aligned}$$

When $a^f = 0$, $\xi^f = 1$, the pendulum is brought to the state $\theta_e = \theta_*(\omega^*)$. Application of the asymptotic methods of [1–3] to problem (3.2) meets with considerable difficulties, particularly in carrying out the averaging operation. To solve this problem one resort to numerical methods for specific values of ε, κ, a^{0f}, ξ^{0f}, $u_{1,2}$. One may also use approximate quasi-optimal control methods corresponding to various relationships among the parameters.

For example, in the case of a strong inequality

$$|\Delta a| \ll |\Delta\xi| \quad (\Delta a = a^f - a^0,\ \Delta\xi = \xi^f - \xi^0)$$

it is natural at the first stage of the control $0 \leqslant \tau \leqslant \tau_*$ to set $u_* = 1/2u^+ + 1/2u^-$ sign $\Delta\xi$, which in a time $\tau_* = |\Delta\xi|\,|\varepsilon u_*|^{-1}$ will bring system (3.12) to the state $\xi = \xi^f$; the quantity a will then vary from a^0 to $a_* = a^0(\xi^0/\xi^f)^{1/2}$, that is, it will be multiplied by $(\xi^0/\xi^f)^{1/2}$. At the final stage $\tau_* < \tau \leqslant \tau_f$, one applies periodic discontinuous control with zero mean, for example, of the form

$$\begin{aligned}
&u_f = u_0(\operatorname{sign} h - \langle\operatorname{sign} h\rangle)\operatorname{sign}(\langle|h|\rangle - \langle h\rangle\langle\operatorname{sign} h\rangle)\operatorname{sign}(a - a^f)\\
&\langle\operatorname{sign} h\rangle = (2/\pi)(\pi - \arccos z^*),\ z^* = -2\varkappa a(a^2 + \varkappa^2)^{-1} \qquad (3.13)\\
&\pi/2 \leqslant \arccos z^* \leqslant \pi,\ \langle h\rangle = a/2,\ u_0 = \min(|u_1|, u_2)
\end{aligned}$$

Application of the control u_f (3.13) over the interval $\tau_* < \tau \leqslant \tau_f$ makes the value of a vary form a^* to the required value a^f, obeying the equation

$$a^{\bullet} = -\varepsilon(u_0/\xi^f)\,|\langle|h|\rangle - \langle\operatorname{sign} h\rangle\langle h\rangle|\operatorname{sign}(a - a^f) \qquad (3.14)$$

Since u_f has zero mean as a function of ψ, this implies $\xi(\tau) = \xi^f + O(\varepsilon)$. The coefficient of sign$(a - a^f)$ in (3.14) is strictly negative and for $\kappa > 0$ it has an upper bound. Hence, in a finite time

$$\tau_f - \tau_* \sim |a_* - a^f|\,\xi^f(\varepsilon u_0 h_f)^{-1}$$

the amplitude a will take the given value a^f. The coefficient h_f is positive for $\kappa > 0$ and in the case of small a we have the estimate $h_f = 2\,\kappa/\pi + O(a)$. The expression $\langle|h|\rangle$ may be obtained in explicit analytical form by quadratures, on the basis of the known roots ψ_i of the function h, but it is too cumbersome to be presented here. In addition, the right-hand side of Eq. (3.14) depends only on a, this is, it can be integrated by quadratures.

Note that the control made (3.13), (3.14) for system (3.12) is also applicable with general assumptions concerning the quantities Δa and $\Delta\xi$. However, for large $|\Delta a|$ it may lead to a very inferior performance index τ_f, since the presence of the initial stage, in which the quantity ξ decreases substantially when a varies in the prescribed manner, may prove to be more effective. the purpose of this variation of ξ is to increase the coefficient of the control u for a. This mode can also contain three stages: (1) a decrease in ξ; (2) the required variation of the variable a, and (3) bringing the variable ξ to the value ξ^f. These stages are implemented separately by a procedure analogous to (3.13), (3.14). Note that decreasing the parameter ξ leads to an increase in the amplitude a, $a \sim \xi^{-1/2}$, but the effectiveness of varying a increases as ξ^{-1} (see (3.12)), which leads to a considerable gain in time if $|\Delta a| \gg |\Delta\xi|$.

Analysis of Eqs (3.12) indicates that

$$a^{\bullet} = -\varepsilon\varkappa\xi^{-1}u\cos\psi + O(\varepsilon a), \quad a \ll 1, \quad \varkappa \sim 1, \quad \xi^{\bullet} = \varepsilon u$$
$$a^{\bullet} = -\varepsilon\xi^{-1}ua(\cos\psi - \varkappa\sin\psi)\cos\psi + O(\varepsilon\varkappa), \quad \varkappa \sim 1 \ll a \tag{3.15}$$

Controlled systems (3.15) can also tackled by the previously mentioned asymptotic methods for constructing an approximate optimal control [1–3]. To summarize the foregoing arguments: we have constructed precision controls that bring the pendulum to the required state. Along with the time-response functional, one can also consider other performance indices, constraints and final conditions [1–3].

Bringing the pendulum into the neighbourhood of a prescribed state. The above modes of high-precision control in a small neighbourhood of a prescribed state assume that the phase point of the system has first been brought into the relevant domain of values (see (3.6) and (3.11)). This controlled process is conveniently realized by transforming to "energy-phase" variables (E, ψ). Differentiating the expression for E (2.1) along the orbits of Eqs (1.5), we obtain

$$\dot{E} = H(\theta, \dot{\theta}, \omega)u, \quad H \equiv \varkappa\dot{\theta}\cos\theta + \omega\sin^2\theta, \quad \dot{\omega} = u \tag{3.16}$$

The equation for the phase Ψ is determined by quadratures using standard techniques [2, 3]; its explicit form is not essential for what follows. In the oscillatory mode $U_*(\omega^2) \leqslant E \leqslant 2$ (see (2.3)), the stationary point $\dot{\theta} = 0$, $\theta_e = 0$ or $\theta_e = \theta_*(\omega^2)$ corresponds to $E_* = 0$ or $E_* = U_*(\omega^2)$ (for $\omega^2 \leqslant 1$ or $\omega^2 > 1$). System (3.16), (1.5) can be steered to a neighbourhood of the stationary point by a locally optimal control [1] based on the "penalty" method

$$J[u] = (E_* - E(T))^2 + k^2(\omega_* - \omega(T))^2 \to \min_u, \quad u_1 \leqslant u \leqslant u_2 \tag{3.17}$$
$$u^* = \tfrac{1}{2}u^+ + \tfrac{1}{2}u^- \operatorname{sign}[(E_* - E)H(\theta, \dot{\theta}, \omega) + k^2(\omega_* - \omega)]$$

where k^2 is a weighting factor and T is an explicitly or indirectly given sufficiently long time for steering the system into the neighbourhood of the prescribed value of E_*, ω_*. It is assumed that the energy E is computed from (2.1) on the basis of measurements of the quantities θ, $\dot{\theta}$, ω. The coefficient $k^2 \sim 1$ is chosen by mathematical modelling of the control process after substituting the function u^* (3.17) into system (1.5). One can use the control u^* without taking into account an additional relationship between E_* and ω_*, for example, one can bring the pendulum to a state of relative rotation: $E_* > 2$. Conversely, by (3.17), the pendulum is brought from a rotational mode to a certain neighbourhood of the desired state of oscillations. Controlled oscillations and rotations with $u \sim \varepsilon$ may be investigated using the averaging method for Eq. (3.16).

Control of the rotations of the pendulum. The asymptotic approach makes it comparatively easy to investigate relative controlled rotations of the pendulum in the case rapid rotations

$$|\dot{\theta}| \gg |\omega| \sim 1$$

We will briefly present a suitable procedure for introducing a small parameter and constructing a controlled system of standard form. Let $\dot{\theta} = \Lambda\sigma$, where $\Lambda \gg 1$ is a constant characterizing the velocity of rotations in dimensionless time $t' = \nu t$, for example $\Lambda = \dot{\theta}$, and $\sigma \sim 1$ is an unknown variable. Then, introducing a new argument $\tau = \Lambda t'$—the fast phase—we reduce Eqs. (1.5) (with $\varepsilon = 1$) to the desired form [2, 3]

$$\sigma^{\bullet} = -\mu\varkappa u\cos\theta - \mu\sin\theta + \mu\omega^2\sin\theta\cos\theta \tag{3.18}$$
$$\theta^{\bullet} = \sigma, \quad \omega^{\bullet} = \mu u, \quad u_1 \leqslant u \leqslant u_2, \quad \mu = \Lambda^{-1} \ll 1$$

where μ has the meaning of a small parameter, θ is the phase, and the control $u \sim 1$; the dot denotes differentiation with respect to τ.

Considering the controlled system (3.18), one can formulate and approximately solve a time-optimal control problem of type (3.7) with the following changes in the notation

$$t \to \tau, \quad t_f \to \tau_f, \quad \varepsilon \to \mu, \quad a \to \sigma, \quad \gamma \to \omega, \quad \Omega \to 1$$

A solution with relative error $O(\mu)$ is given by formulae (3.9) and the graphs of Fig. 4. The optimal control $u^*(\theta, \sigma, \omega)$ brings the phase point of system (3.18) into a μ-neighbourhood of the desired value (σ^f, ω^f) in the interval $0 \leqslant \tau \leqslant \tau_f \sim \mu^{-1}$. The corresponding solution differs by a small quantity $O(\mu)$ from the exact solution with respect to the slow variable and the time of the process.

Control of the relative oscillations and rotations of the pendulum in the case of zero "arm". An important quantity in the control problems investigated above was κ ($\kappa \sim 1$). If $\kappa \ll 1$, the problems may be formulated differently in the limit when $\kappa = 0$. To that end, we express the non-dimensionalized equations of motion (1.5) in the form

$$\ddot{\theta} + \sin\theta = \varepsilon u \sin\theta\cos\theta, \quad 0 \leqslant u \leqslant 1, \quad \varepsilon = \omega_0^2 \ll 1$$
$$\dot{\varphi} = \sqrt{\varepsilon} v, \quad -1 \leqslant v \leqslant 1, \quad u = v^2, \quad \varphi(\mathrm{mod}\, 2\pi) \tag{3.19}$$

According to Eqs (3.19), it is assumed that the velocity of rotation of the base changes virtually instantaneously and may take values within the limits

$$-\omega_0 \leqslant \dot{\varphi} = \omega_0, \quad \omega_0 = \sqrt{\varepsilon}$$

Considering system (3.19), one can formulate and solve problems in which the energy of relative oscillations and/or rotations of the pendulum and the orientation of the base are to be varied in an optimal manner over an asymptotically long time interval. Discontinuous changes in the velocity of rotation of the base may be implemented by an electric motor.

Let us reduce Eq. (3.19) for θ to the standard form [1–3]

$$\dot{E} = \varepsilon u \dot{\theta} \sin\theta\cos\theta, \quad \dot{\Psi} = \frac{2\pi}{T(E)} + \varepsilon u F(E, \theta)$$
$$E = \frac{1}{2}\dot{\theta}^2 - \cos\theta \geqslant -1, \quad T(E) = \oint \frac{d\theta}{\dot{\theta}(\theta, E)}, \quad \dot{\theta} = \pm\sqrt{2}(E + \cos\theta)^{1/2} \tag{3.20}$$

The explicit form of the function F is unimportant. Let us consider the problem of the time-optimal variation of the energy E ignoring changes in the phase Ψ according to (3.20) and in the angle φ (3.19). Then, after changing E to the desired value E^f with error $O(\varepsilon)$ in time $t_f \sim \varepsilon^{-1}$, by applying the constant control $v = \pm 1$, $u = 1$, the angular variable can also be brought to the desired value $\varphi_f(\mathrm{mod}\, 2\pi)$ in a relatively short time $\nabla t_f \sim \varepsilon^{-1/2}$, in such a way that

$$E = E^f + O(\varepsilon), \quad t_f < t \leqslant t_f + \Delta t_f$$

The optimal feedback control u^*, the optimal variation of E and other characteristics have the form

$$u^* = \tfrac{1}{2} + \tfrac{1}{2}\operatorname{sign}(\dot{\theta}\sin\theta\cos\theta)\operatorname{sign}(E^f - E)$$
$$dE/d\tau = H(E)\operatorname{sign}(E^f - E), \quad \tau = \varepsilon t$$
$$H(E) = \frac{1}{2T}\int_0^T |\dot{\theta}\sin\theta\cos\theta|\, dt = \frac{1}{2T}\oint |\sin\theta\cos\theta| \operatorname{sign}\dot{\theta}\, d\theta \tag{3.21}$$
$$\tau = \operatorname{sign}(E^f - E^\circ)\int_{E^\circ}^{E} \frac{d\xi}{H(\xi)}, \quad \tau_f = \left|\int_{E^\circ}^{E^f} \frac{dE}{H(E)}\right|$$

The analytical expressions for the functions $H(E)$ and $T(E)$ depend on the mode of motion of the pendulum. In oscillatory or rotatory motion, we have the following explicit representations for $H_{v,r}$ and $T_{v,r}$ [1–3]

$$H_v(E) = (1 \pm E^2)/T_v(E), \quad 0 \lessgtr E, \quad T_v = 4\mathbf{K}(k_v), \quad k_v^2 = \tfrac{1}{2}(E+1), \quad |E| < 1 \tag{3.22}$$
$$H_r(E) = 1/T_r(E), \quad T_r(E) = 2k_r\mathbf{K}(k_r), \quad k_r^2 = 2/(E+1), \quad E > 1$$

where $\mathbf{K}(k)$ is the complete elliptic integral of the first kind with modulus k, $k^2 < 1$.

It follows from (3.22) that the effectiveness of the control u^* in rotary motion increases as E increases, since

$$T_r(E) \approx 2\pi/\sqrt{2E}, \quad E \to \infty$$

In oscillatory motion $(-1 < E < 1)$, the effectiveness of the control decreases as $E \downarrow -1$, namely: $E^* \approx (-1 + E)/\pi$, which leads to an unlimited increase in the time for "entering" the state of "leaving" the state of rest of the pendulum $(E^{f,0} = -1)$, in accordance with the estimate $\tau \sim |\ln(E + 1)|$. The effectiveness also decreases as $E \uparrow 1$, since $T_v \to \infty$ in accordance with the estimate

$$T_v \approx 4\ln(2\sqrt{2}/\sqrt{1-E}) \sim |\ln(1-E)|$$

However, this logarithmic singularity is integrable, so that one has a finite time for "crossing" the separatric $(E = 1)$. Similarly, if E decreases in oscillatory motion $(E > 1)$, passage through the value $E = 1$ also requires a finite time, since the expression for $T_r(E)$ has an analogous logarithmic asymptotic form.

Although the motions are no longer periodic near the separatrices and this makes it difficult to use the averaging method, the error thus incurred, due to the "scattering property" of the separatrix, implies an asymptotically small error $O(\varepsilon|\ln\varepsilon|)$ [7], which is acceptable in the sense of the required accuracy, provided that $\varepsilon > 0$ is sufficiently small.

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